Lecture 4. Dimensional analysis
Dimensionless numbers in fluid dynamics

Last time: we derived the Navier-Stokes equations for incompressible flows without gravity

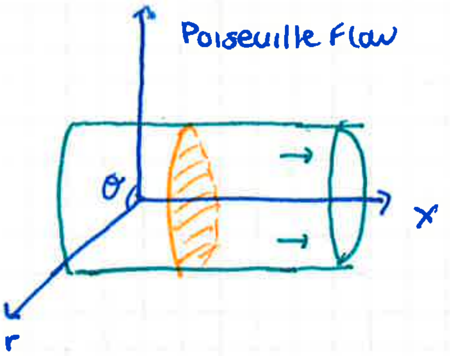
$$\nabla \cdot \underline{v} = 0$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \underline{v}$$

Navier-Stokes equations for incompressible flows where ρ = density of fluid (constant), $\nu = \frac{\eta}{\rho}$ kinematic viscosity

These equations are the basis of a vast part of fluid dynamics of practical interest. The non-linear term in the second equation results in even very simple set ups having rich behavior.

Let's consider the simplest example we can think of: flow through a pipe.



We'll consider a unidirectional steady (incompressible) flow through a pipe. In cylindrical coordinates we can write the velocity as:

$$\underline{v} = v_x \hat{e}_x$$

We'll assume that $v_x = v_x(x, r)$, i.e. does not depend on θ, since the boundary is circular.

The condition for incompressibility implies:

$$\nabla \cdot \underline{v} = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(\cancel{r v_r})^{0} + \frac{1}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}}^{0} + \frac{\partial v_x}{\partial x} = 0 \Rightarrow v_x \text{ is not a function of } x$$

$$\Rightarrow v_x = v_x(r)$$

impermeable walls — no net radial flow

So we have that: $\underline{v} = v_x(r)\hat{e}_x$ from the incompressibility condition. For the non-linear term in N-S:

$$\underline{v} \cdot \nabla = v_x \frac{\partial}{\partial x} + \cancel{v_r \frac{\partial}{\partial r}}^{0} + \cancel{v_\theta \frac{1}{r}\frac{\partial}{\partial \theta}}^{0}$$

(Remember in cylindrical coord $\nabla = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}$)

$$\Rightarrow \underline{v} \cdot \nabla v = v_x \frac{\partial}{\partial x}(v_x(r)\hat{e}_x) = 0$$

since v_x does NOT depend on x (as found by incompressibility condition)

~~Solv~~ So N-S reduces to (Remembering steady flow $\frac{\partial v}{\partial t} = 0$)

$$0 = -\frac{\nabla p}{\rho} + \nu \nabla^2 \underline{v} \Leftrightarrow 0 = -\nabla p + \eta \nabla^2 \underline{v}$$

We can write out each component of this equation remembering the expression for the Laplacian in cylindrical coordinates:

$$\nabla^2 \underline{v} = \hat{e}_x(\nabla^2 v_x) + \hat{e}_r\left[\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \hat{e}_\theta\left[\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right]$$

$$\left(\text{where } \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right)$$

In our case the Laplacian reduces to:

$$\nabla^2 \underline{v} = \hat{e}_x \nabla^2 v_x = \hat{e}_x \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right)$$

So we can write the three components of N-S as:

$$\left.\begin{aligned} -\frac{\partial p}{\partial r} &= 0 \\ -\frac{1}{r}\frac{\partial p}{\partial \theta} &= 0 \end{aligned}\right] \Rightarrow p = p(x)$$

$$-\frac{\partial p}{\partial x} + \eta \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) = 0$$

since v_x only depends on r and p only depends on x

$$\Rightarrow \frac{\partial p}{\partial x} = -G \text{ , where } G \text{ is a constant}$$

$$\Rightarrow p = p_0 - Gx \quad \text{with } G \equiv \text{constant}$$

integrating the above equation

So our equation for v_x becomes:

$$G + \eta \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) = 0$$

$$\Rightarrow -\frac{G}{\eta} r = \frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) \Rightarrow \frac{dv_x}{dr} = -\frac{Gr}{2\eta} + \frac{A}{r} \Rightarrow v_x(r) = -\frac{Gr^2}{4\eta} + \cancel{A\ln(r)} + B$$

$A=0$ so the velocity does not diverge at $r=0$

$$\Rightarrow v_x(r) = \frac{-Gr^2}{4\eta} + B$$ where we can determine B considering a no-slip BC ($v_x(r=a) = 0$)

$$\Rightarrow 0 = -\frac{Ga^2}{4\eta} + B \Rightarrow \boxed{v_x = \frac{G}{4\eta}(a^2 - r^2)}$$

This all seem nice, however this flow becomes unstable (turbulent) at Reynolds number $\sim 3{,}000$

($Re \equiv \frac{UL}{\nu}$ U velocity L lengthscale ν viscosity)

We could've seen this coming, if we looked at the dimensionless N-S equations:

First we can define dimensionless spatial and time variables:

$$\tilde{t} = \frac{tU}{L} \; ; \; \tilde{\underline{r}} = \frac{\underline{r}}{L} \; ; \; \tilde{\underline{v}} = \frac{\underline{v}}{U}$$

In these scaled coordinates, the N-S equations become:

$$\tilde{\nabla}\cdot\tilde{\underline{v}} = 0$$

$$\frac{UL}{\nu}\left[\frac{\partial \tilde{\underline{v}}}{\partial \tilde{t}} + \tilde{\underline{v}}\cdot\tilde{\nabla}\tilde{\underline{v}}\right] = -\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2\tilde{\underline{v}}$$

where $Re \equiv \frac{UL}{\nu}$ The Reynold's number

When $Re \gg 1$ we cannot ignore the left side of this equation anymore. In essence the Re measures the relative importance of the non-linear term $\underline{v}\cdot\nabla\underline{v}$ in N-S.

We will encounter other dimensionless numbers in different situations. Typically a dimensionless number is the ratio of one physical effect over another, so that its magnitude is an indication of which effects are the dominant ones for each situation. This is [illegible] one of the reasons why dimensional analysis becomes important in fluid dynamics.

Dimensional analysis in fluid dynamics helps reduce the complexity of the governing equations by grouping variables into non-dimensional groups/parameters. This not only reveals fundamental physics driving the flow but allows us to generalize experimental results across lengthscales w/o solving equations all the time. (dynamic similarity)

Buckingham pi theorem

Problems in fluid dynamics often involve a large number of variables w/different dimensions. The buckingham pi theorem provides a systematic method for reducing such problems by expressing them in terms of a smaller number of independent dimensionless parameters.

"If a physically meaningful equation involves "n" variables with dimensions, and the variables are expressed in terms of K fundamental dimensions (e.g. mass, length, time), then the original equation can be written in terms of a set of $p = n-k$ dimensionless parameters known as π terms"

If a physical phenomenon involves N quantities $A_1, A_2 ..., A_N$. That are related by:

$$f(A_1, A_2 ..., A_n) = 0$$

The π theorem states that the equation

$$f(A_1, ... A_n) = 0$$

can be rewritten as:

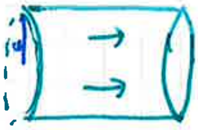
$$F(\pi_1, \pi_2 .., \pi_{n-k}) = 0$$

where π_i are dimensionless, independent parameters. These dimensionless quantities can be constructed as:

$$\pi_i = \frac{A_{k+i}}{A_1^{\beta_i(1)} A_2^{\beta_i(2)} \dots A_k^{\beta_i(n)}}$$

where $\beta_i^{(j)}$ are the exponents needed such that π_i is dimensionless.

We will see how this theorem works by applying it to Poiseuille flow.



Flow through a pipe of radius a of a fluid w/density ρ and viscosity η
Driven by a pressure gradient $\nabla p = -G$

We want to get information on the velocity field $\underline{V} = V_x \hat{e}_x$
What are the relevant physical quantities of the problem?

$$V_x, r, \cancel{\theta}, \cancel{x}, a, G, \eta, \cancel{\rho}$$

θ, x drop out due to symmetry of the problem ; ρ is not considered if we neglect inertial effects.

so we have:

$$V_x, r, a, G, \eta \Rightarrow n = 5$$

so the solution has the form:

$$F(V_x, r, a, G, \eta) = 0$$

What are the units of my quantities?

$$[V_x] = l\,t^{-1}$$

$$[r] = [a] = l$$

$$[\eta] = \frac{[\sigma]}{[\dot{\gamma}]} = \frac{F/A}{t^{-1}} = \frac{\frac{ml}{t^2 l^2}}{t^{-1}} = \frac{m}{lt}$$

So we have 3 fundamental dimensions that all quantities depend on, ($k=3$)

$$[G] = [\nabla][p] = \frac{1}{l}\frac{F}{A} = \frac{m}{l^2 t^2}$$

We will take the quantities a, η, G and construct with them the π terms that involve V_x, r. We chose those quantities since we should not use the quantity of interest (i.e V_x) or the independent variables (i.e r) to construct pi terms.

Then we'll construct Π terms by:

$$\Pi_1 = \frac{V_x}{a^{\alpha_1}\eta^{\alpha_2}G^{\alpha_3}}$$

$$\Pi_2 = \frac{r}{a^{\beta_1}\eta^{\beta_2}G^{\beta_3}}$$

We want to determine the exponents $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ such that Π_1 & Π_2 are dimensionless.

For Π_2 this is straightforward since:
$[r] = l$ so we can choose $\beta_1 = 1$, $\beta_2 = \beta_3 = 0$

so $\Pi_2 = \dfrac{r}{a}$

For Π_2:

$$[V_x] = [a^{\alpha_1}][\eta^{\alpha_2}][G^{\alpha_3}]$$

$$\Rightarrow \quad l t^{-1} = l^{\alpha_1} m^{\alpha_2} l^{-\alpha_2} t^{-\alpha_2} m^{\alpha_3} l^{-2\alpha_3} t^{-2\alpha_3}$$

so we get a system of algebraic equations for the exponents:

For l:
$$1 = \alpha_1 - \alpha_2 - 2\alpha_3$$

For t:
$$-1 = -\alpha_2 - 2\alpha_3$$

~~For~~ For m:
$$0 = \alpha_2 + \alpha_3$$

$$1 = \alpha_1 + 1 - 2 \Rightarrow \alpha_1 = 2$$

$$-1 = +\alpha_3 - 2\alpha_3 \Rightarrow \alpha_3 = 1$$

$$\Rightarrow \alpha_2 = -\alpha_3 \quad \Rightarrow \alpha_2 = -1$$

With this our Π terms become:

$$\Pi_1 = \frac{V_x}{a^2\eta^{-1}G} \quad ; \quad \Pi_2 = \frac{r}{a}$$

I can construct the equation using these pi terms:

$$F(\Pi_1, \Pi_2) = 0 \Rightarrow \Pi_1 = g(\Pi_2)$$ Π_1 is a function of Π_2

$$\Rightarrow \frac{V_x}{a^2\eta^{-1}G} = g\left(\frac{r}{a}\right)$$

$$\Rightarrow V_x = \underbrace{\frac{a^2 G}{\eta}}_{\text{dimensional prefactor}} \underbrace{g\left(\frac{r}{a}\right)}_{\text{dimensionless}}$$

dimensional prefactor → sets the scale of the velocity

dimensionless → actual shape of velocity profile as a function of r

and from the solution we got by directly solving N-S we know that

$$V_x = \frac{G}{4\eta}(a^2 - r^2)$$

$$\Rightarrow g\left(\frac{r}{a}\right) = \frac{1}{4}\left[1 - \left(\frac{r}{a}\right)^2\right]$$

Another example

Consider the steady flow of water around a boat hull. We want to know what is the form of the drag force, F_D, on the hull.

The drag force on the hull arises due to viscous stresses that water exerts on the hull as it moves plus pressure forces that arise from the surface waves produced by the motion of the hull (gravity waves).

The drag force in question is the surface integral of the stress over the hull:

$$F_D = \int_{hull} \underline{\underline{\sigma}} \cdot \hat{n}\, dS$$

An alternative approach to find Π terms is to define the characteristic scales of the problem, define non-dimensional variables and write the dimensionless equation of motion to see what dimensionless groups appear.

- Defining the characteristic scales of the problem:

$L \equiv$ characteristic lengthscale
$U \equiv$ characteristic velocity scale
$\frac{L}{U} \equiv$ characteristic timescale

$\rho U^2 \equiv$ characteristic pressure scale $\left(\rho \frac{D\underline{v}}{Dt} \propto \frac{\rho U^2}{L} \Rightarrow p \propto \frac{\rho V^2}{L} \Rightarrow \nabla p \propto \rho V^2 \right)$

With this we define the dimensionless variables:

$$\underline{x}' = \frac{\underline{x}}{L} \quad ; \quad \underline{v}' = \frac{\underline{v}}{U} \quad ; \quad t' = \frac{tU}{L} \quad ; \quad p' = \frac{p}{\rho U^2}$$

Our equation of motion is the N-S equation for incompressible flow w/ gravity:

$$\rho \frac{D\underline{v}}{Dt} = -\nabla p + \eta \nabla^2 \underline{v} + \rho \underline{g}$$

In terms of dimensionless variables this becomes:

$$\frac{D}{Dt'} \underline{v}' = -\nabla' p + \frac{\eta}{UL\rho} \nabla'^2 \underline{v}' - \frac{gL}{U^2} \hat{e}_z \quad (*)$$

Now, defining the following dimensionless groups:

$$Re \equiv \frac{UL\rho}{\eta} = \frac{UL}{\nu} \quad ; \quad Fr \equiv \frac{U}{\sqrt{gL}}$$

Reynold's number: compares viscous forces $\propto \frac{\mu V}{L^2}$ to inertial forces $\propto \frac{U^2 \rho}{L}$

Froude number: compares inertial forces $\propto \frac{U^2 \rho}{L}$ to gravitational forces $\propto \rho g$

$\Bigg($ The inertial term in N-S: $\rho \frac{D\underline{v}}{Dt} = \rho \left[\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right]$
$\Rightarrow \rho \frac{\partial}{\partial t} \underline{v} \;\; \mathcal{O} \left(\frac{\rho U}{L/U} \right)$
$\rho \underline{v} \cdot \nabla \underline{v} \;\; \mathcal{O} \left(\rho U \frac{U}{L} \right)$
$\Bigg|$ The viscous term: $\eta \nabla^2 \underline{v} \;\; \mathcal{O} \left(\eta \frac{U}{L^2} \right) \Bigg)$

With this our equation of motion becomes:

$$\frac{D\underline{v}'}{Dt} = -\nabla' p' + \frac{1}{Re} \nabla'^2 \underline{v}' + \frac{1}{Fr^2} \hat{e}_z$$

From this I get Re and Fr as Π groups. How many do I need? The drag ~~drag~~ may depend on:
F_D, ρ, U, L, ν, g ; the fundamental dimensions are m, ℓ, t $\Rightarrow k = 6 - 3 = 3$

I already have Re and Fr, just need to construct an additional Π term.

This last pi term will be the drag coefficient; the norm of the dimensionless drag force:

$$C_D \doteq \frac{|F_D|}{\rho U^2 L^2} \qquad \left([F_D] = [\underline{\underline{\sigma}}][A] = \rho U^2 L^2 \right)$$

So I just need to make sure that my Π terms are independent, that is, I cannot write one as a combination of other. One easy way ~~to ensure they are~~ to ensure they are all independent is to incorporate a new variable into each pi term. In this case we have

$\Pi_1 = Re = \frac{UL}{\nu}$; $\Pi_3 = C_D = \frac{|F_D|}{\rho U^2 L^2}$ all contain a new variable so are independent.

$\Pi_2 = Fr = \frac{U}{\sqrt{gL}}$

According to the buckingham pi theorem we can write

$$F(\Pi_1, \Pi_2, \Pi_3) = 0$$

$$\Rightarrow \Pi_3 = g(\Pi_1, \Pi_2) \quad \text{so we have} \quad C_D = g(Re, Fr)$$

We can perform dimension analysis on any fluid mechanics problem and create a set of dimensionless groups (Π terms) but they might be different from the ones that are commonly defined. Dimensionless groups are refered to as numbers and have a physical interpretation. Here is a table with some key dimensionless parameters of fluid dynamics relevant to soft matter.

- Re $= UL/\nu$, the **Reynolds number**. Well-known dimensionless number defined in equation (1.52). Expresses the ratio of inertial forces to viscous forces, for flows characterized by a velocity U around a body of size L. ν is the kinematic viscosity defined in (1.36). See section 1.11 for discussion of Re dependence of flows.
- Pe $= \dot{\gamma} R^2/D$, the **Péclet number**. Expresses the competition between advection and diffusion in a shear gradient $\dot{\gamma}$ for particles of radius R and with diffusion coefficient D. Introduced in equation (4.25) in our discussion of colloidal rheology.
- Pr $= \nu/D_T$, the **Prandtl number**. As the ratio of the kinematic viscosity ν over the thermal conductivity D_T it expresses the competition between viscous damping and thermal diffusion. In typical simple fluids, Pr is just somewhat larger than 1, expressing that momentum diffusion is a bit faster than thermal diffusion. Plays a role in Rayleigh-Bénard convection; see section 8.3.1.
- Ra $= \alpha g d^3 \Delta T/\nu D_T$, the **Rayleigh number**. Measures the importance of buoyancy effects induced by a thermal gradient, relative to stabilizing viscous damping and thermal diffusion. As discussed in section 8.3.1 on Rayleigh-Bénard convection, α measures the thermal expansion, g is the gravitational acceleration, d the thickness of the Rayleigh-Bénard cell, and ΔT the temperature across it. Defined in equation (8.11).
- Sc $= \nu/D$, the **Schmidt number**. As molecular diffusion, characterized by the diffusion coefficient D, is typically slower than momentum or thermal diffusion, the Schmidt number is large, typically of order 10^3, for simple fluids. This disparity of the time scales of viscous damping and molecular diffusion gives rise to many physicochemical hydrodynamic effects, but it complicates numerical and experimental studies.[45]
- Le $= D_T/D$, the **Lewis number**. Important when both molecular and thermal diffusion play a role. As Le $=$ Sc/Pr, the Lewis number is typically large.
- Ca $= \eta U/\gamma$, the **capillary number**. With η the dynamic viscosity and γ the surface tension, the capillary number measures the ratio of viscous to capillary forces. See e.g. the wetting front analysis in problem 7.2.
- Bo $= \rho g R^2/\gamma$, the **Bond number**. Measures the ratio of gravitational to capillary forces for a droplet of radius R and density ρ. Note that Bo $= R^2/\ell_c^2$, with ℓ_c the capillary length defined in equation (1.63). The change of shape with radius R of droplets on a leaf in figure 1.21 is due to the varying Bond number.
- We $= \rho U^2 R/\gamma$, the **Weber number**. Measures the ratio of inertial to capillary forces, e.g., during the deformation of a droplet of radius R hitting a surface with velocity U.
- Ma $= R\Delta\gamma/\rho\nu D$, the **Marangoni number**. Compares the Marangoni forces discussed in section 1.13.3 with the stabilizing viscous forces and stabilizing mass diffusion. Note that the surface tension difference $\Delta\gamma$ can arise from concentration gradients and thermal gradients.
- Oh $= \eta/(\rho\gamma R)^{1/2} = \text{We}^{1/2}/\text{Re}$, the **Ohnesorge number**. Gives the ratio of the time scale of viscous damping to the time scale of capillary oscillations.
- Wi $= |\sigma_{xx} - \sigma_{yy}|/\sigma_{xy}$, the **Weissenberg number**. As discussed in section 5.10.2, for viscoelastic polymer fluids the Weissenberg number is a ratio of the (first) normal stress difference over the shear stress for a simple shear flow. In the UCM model (5.80) for polymer rheology, Wi $= 2\dot{\gamma}\lambda$, where $\dot{\gamma}$ is the shear rate and λ is the relaxation time of the fluid; see equation (5.81). This ratio of the relaxation time to the time scale of the flow in polymer flows is also called the Deborah number De.

1.17 Box 1: Key dimensionless parameters

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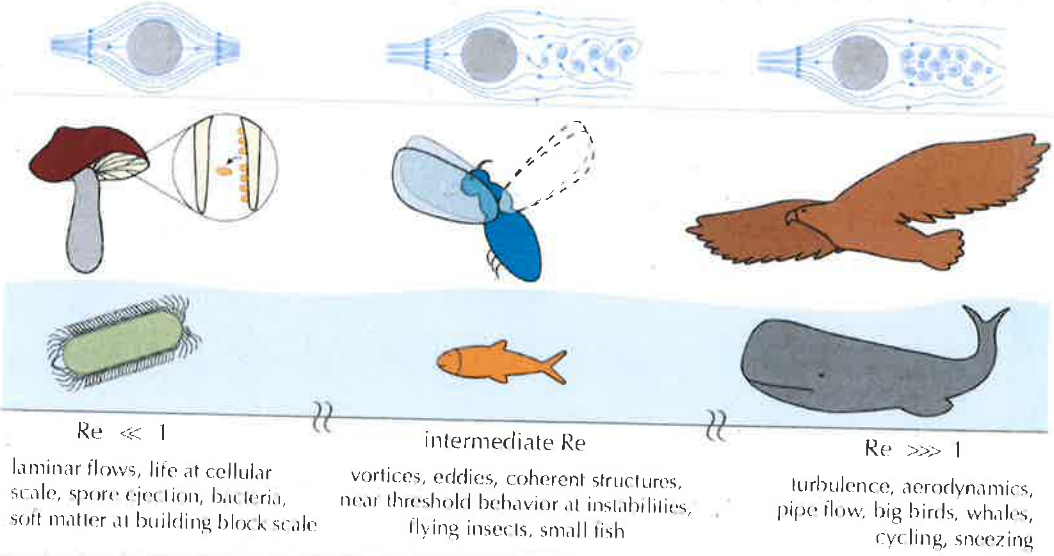
- The **Damköhler number** Da is in the presence of chemical reactions defined as the ratio of the chemical reaction rate and the diffusive or convective mass transport rate.

From small to large Reynold's numbers

We saw at the beginning of the lecture that pressure-driven flow in a pipe can display radically different physical regimes for increasing fluid speeds, from smooth and laminar to unpredictable & turbulent. The Reynold's number is the one number that controls the transition to turbulence. The Reynold's number is the one parameter governing the dynamics of viscous fluids.



$$Re \equiv \frac{UL}{\nu} \sim \frac{\text{inertial F's}}{\text{viscous F's}}$$

U characteristic speed
L characteristic length
ν viscosity

$$\frac{D\underline{v}'}{Dt'} = -\nabla' p + \frac{1}{Re}\nabla'^2 \underline{v}'$$

For steady flow this reduces to:

$$\underline{v}' \cdot \nabla' \underline{v} = -\nabla' p' + \frac{1}{Re}\nabla'^2 \underline{v}^2 \quad (*)$$

Low Reynold's number

When $Re \ll 1$ viscous forces dominate over inertia so much that we can neglect the role of inertia. So dropping the inertial term in N-S equation (*) we are left with

$$0 = -\nabla' p + \frac{1}{Re}\nabla'^2 \underline{v}^2 \quad \text{or in dimensional form} \quad \boxed{\begin{aligned}\nabla p &= \nu \nabla^2 \underline{v} \\ \nabla \cdot \underline{v} &= 0\end{aligned}}$$

at every point in the fluid there is a balance between pressure & viscous forces.

Stokes equations for incompressible Newtonian fluid

smooth laminar flow patterns

Intermediate Reynold's number

As the Reynold's number increases, the inertial non-linear term $\underline{v} \cdot \nabla \underline{v}$ becomes more important so there is a transition to more complicated flows. Typical scenario:

Flow past a cylinder

$Re \ll 1$

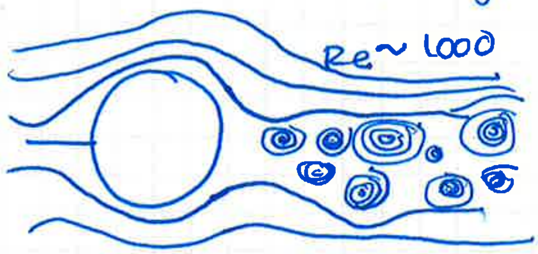
$Re \sim 10$
Coherent vortices form, but stay behind the cylinder

$Re \sim 100$
Vortices are being shed from the cylinder.

Structures and patterns in this regime are still quite coherent

Very large Reynolds number

In this regime we have turbulence, flows become highly irregular and chaotic. Turbulence causes the formation of eddies of many different lengthscales.



If we look at our definition of $Re = \frac{\text{inertial}}{\text{viscous}}$ if $Re \gg 1$ we can be tempted to ignore to viscous effects.

If we do this our equation of motion would become:

$$\underline{u}' \cdot \nabla' \underline{u}' = -\nabla' p' \quad \text{or dimensionally} \quad \rho \underline{u} \cdot \nabla \underline{u} = -\nabla p$$

← this is Euler's equation for ideal (inviscid) fluids!

The relationship between this equation and the actual behavior of fluids at $Re \gg 1$ is much more complex that at low Re.

We see that we have discarded the term involving second space derivatives, so we are reducing the order of this differential equation, so we are also reducing the number of boundary conditions. We have discussed that the no-slip BC is a consequence of the action of viscosity, so we have indeed discarded this BC by neglecting the viscous term.

The Euler equation of motion can be solved by using only the impermeability condition ($\underset{\substack{\uparrow \\ \text{vel of fluid at wall}}}{\underline{v}} \cdot \hat{n} = \underset{\substack{\uparrow \\ \text{vel of wall}}}{U} \cdot \hat{n}$)

If we try and impose also the no slip BC finding a solution is not possible! Consequently the viscous term in the dynamical equation must always remain significant in the vicinity of the boundary. The region in which this happens is known as the boundary layer.

Another way to think about it is if we get rid of the viscous term completely, this implies no drag! Which is at odds with what happens in real life no matter how large Re is.

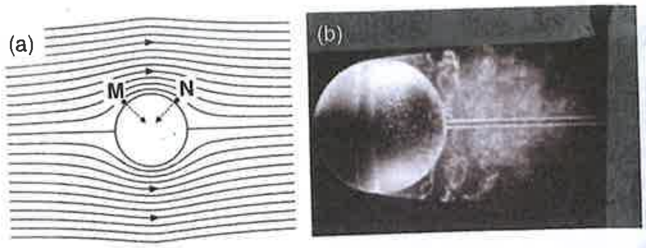
The reasoning that viscous force should be negligible breaks down in the boundary layer because the flow develops an internal lengthscale much smaller than the imposed lengthscale L. This smaller lengthscale is the boundary layer thickness, δ. So in our equations of motion the size of the viscous term is determined by δ while the size of the inertial term is determined by L.

$$|\underline{u} \cdot \nabla \underline{u}| \sim \frac{U^2}{L} \quad |\nu \nabla^2 \underline{u}| \sim \frac{\nu U}{\delta^2}$$

Inertia and viscous forces can remain comparable in magnitude if

$$\frac{U^2}{L} \sim \frac{\nu U}{\delta^2} \Rightarrow \frac{\delta}{L} \sim \left(\frac{UL}{\nu}\right)^{-1/2} = Re^{-1/2}$$

the difference between the 2 lengthscales becomes more marked as Re increases.



25. The flow past a rigid body at high Reynolds numbers was long thought to be similar to that obtained by neglecting viscosity. However, in the flow of fluid with zero viscosity ((a) illustrated on a cylinder), at two points M and N symmetric to one another, the fluid speeds are identical, so from Bernoulli's equation the pressures are the same, and therefore the total drag is zero. Prandtl showed that viscous stresses play a crucial role near rigid surfaces. The flow in (b), occurring at Re = 15,000 around a rigid sphere, is clearly qualitatively different from that in (a).

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